Geometric Modeling
Assignment sheet #3
“Curvature & Approximation”
(due May 22nd 2012 before the lecture)

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Exercise 1 (Curvature): **[2 points]**

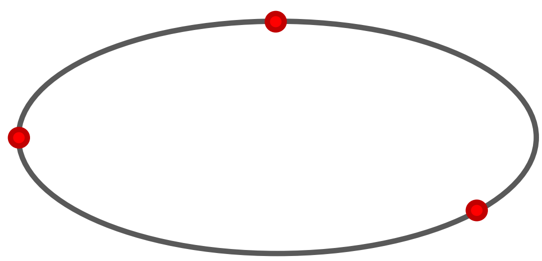
Derive the curvature function $\kappa(t)$ for the following functions:

$$f_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad f_2(t) = \begin{pmatrix} \cos t \\ t \\ \sin t \end{pmatrix}$$

Exercise 2 (Osculating circle): **[2+2 points]**

a. Show that the curvature of a circle with radius r is given by $\pm\frac{1}{r}$.

b. The *evolute* of a curve is the locus of all its centers of curvature (centers of osculating circles). Sketch the evolute for the ellipse below and the osculating circle for the marked points.



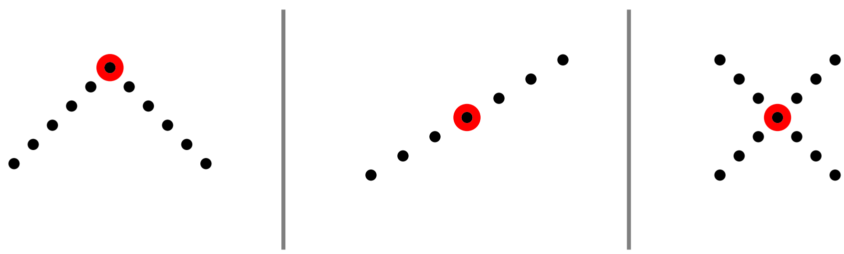
Exercise 3 (Least-Squares Approximation): **[4 points]**

Given n sample points $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \ldots \begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathbb{R}^2$, how do you find the center $\begin{pmatrix} a \\ b \end{pmatrix}$ and radius r of the best fitting circle in an algebraic sense?

Hint: Represent the circle as an implicit function.
Substituting $c = a^2 + b^2 - r^2$ may be helpful to reduce the degree of the error function (this means effectively solving for (a,b,c) instead of (a,b,r).

Exercise 4 (Principal Component Analysis): **[1 point]**

Sketch the normal, tangent and the PCA ellipsoid for the marked points and their neighbors in the following 3 diagrams (no calculations necessary, a rough sketch is sufficient) :



Exercise 5 (Total Least Squares): [7+2 points]

Given n points $\mathbf{p}_1,\ldots,\mathbf{p}_k$ in three dimensional Euclidean space, a best fitting plane in a least-squares sense can be computed by PCA (principal component analysis). The procedure is the following:

- Compute the average $\mathbf{p}_{av} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i$. Subtract this from every point: $\mathbf{d}_i = \mathbf{p}_i - \mathbf{p}_{av}$
- Form the scatter matrix $\mathbf{S} = \mathbf{X}\mathbf{X}^T, \mathbf{X} := \begin{pmatrix} | & & | \\ \mathbf{d}_1 & \ldots & \mathbf{d}_n \\ | & & | \end{pmatrix}$.
- The plane is defined by the average $\mathbf{p}_{av}$ as one point in the plane and the eigenvector of **S** with the smallest eigenvalue as the normal of the plane (we assume this vector to be uniquely determined up to sign and length).

An alternative approach to plane fitting is “plate tensor voting”. The procedure is the following:

- Compute the average as before.
- Compute the vector from the average to each point, denoted by $\mathbf{d}_i$. Let $l_i = ||\mathbf{d}_i||$ be the length of the vector.
- Form the matrices $\mathbf{M}_i = l_i\mathbf{I} - [\mathbf{d}_i]\cdot[\mathbf{d}_i]^{\mathrm{T}}$.
- Average all the matrices $\mathbf{M}_i$: $\mathbf{M} := \frac{1}{n}\sum_{i=1}^{n}\mathbf{M}_i$.
- Compute the normal vector as eigenvector corresponding to the largest eigenvalue of **M**. The plane is defined by this normal vector and the average point (same point as before).

a. Show formally that both schemes compute the exact same solution (i.e., the same plane). You can assume that all eigenvalues are different so that the plane is uniquely defined.

b. Explain in 1-2 sentences what the matrices $\mathbf{M}_i$ represent intuitively (hint: the name of the scheme is derived from this observation). Why do we expect them to sum up to a matrix for which the eigenvector with largest eigenvalue is the normal of the plane? (1-2 sentences are sufficient).